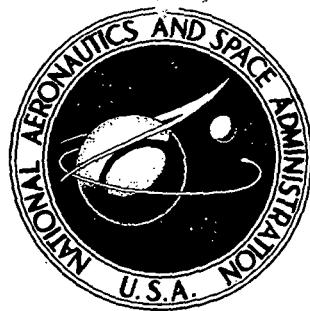


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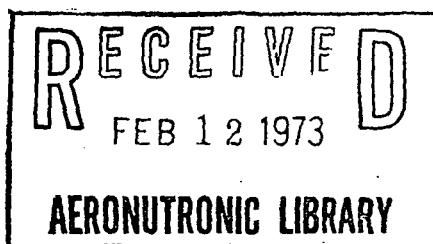


NASA TN D-7101

EXACT ANALYTICAL SOLUTION TO A TRANSIENT CONJUGATE HEAT-TRANSFER PROBLEM

by James Sucec

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1973

1. Report No. NASA TN D-7101	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle EXACT ANALYTICAL SOLUTION TO A TRANSIENT CONJUGATE HEAT-TRANSFER PROBLEM		5. Report Date January 1973	
7. Author(s) James Sucec, University of Maine, Orono, Maine		6. Performing Organization Code	
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135		8. Performing Organization Report No. E-7090	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546		10. Work Unit No. 501-24	
15. Supplementary Notes		11. Contract or Grant No.	
		13. Type of Report and Period Covered Technical Note	
		14. Sponsoring Agency Code	
16. Abstract An exact analytical solution is found for laminar, constant-property, slug flow over a thin plate which is also convectively cooled from below. The solution is found by means of two successive Laplace transformations when a transient in the plate and the fluid is initiated by a step change in the fluid inlet temperature. The exact solution yields the transient fluid-temperature, surface-heat-flux, and surface-temperature distributions. The results of the exact transient solution for the surface heat flux are compared to the quasi-steady values, and a criterion for the validity of the quasi-steady results is found. Also the effect of the plate coupling parameter on the surface heat flux is investigated.			
17. Key Words (Suggested by Author(s)) Transient convection Conjugate Transient		18. Distribution Statement Unclassified - unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 35	22. Price* \$3.00

* For sale by the National Technical Information Service, Springfield, Virginia 22151

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SUMMARY

An exact analytical solution is found for laminar, constant-property, slug flow over a thin plate which is also convectively cooled from below. The solution is found by means of two successive Laplace transformations when a transient is initiated by a step change in the fluid inlet temperature. Two cases are considered: (1) the case where the fluid and the plate are originally at the coolant temperature before the step change in inlet temperature and (2) the case where the fluid and the plate are at the eventual steady state of the first case when another step change in inlet temperature occurs. Results are given for the temperature within the moving fluid as a function of space coordinates and time, for wall surface temperature, and for surface heat flux. The results of the exact solutions are compared with the quasi-steady values, and a criterion for the validity of the quasi-steady solution is evolved. Also the effect of the fluid-to-plate coupling is investigated.

One notable difference between this type of transient (a step change in the fluid inlet temperature) and the transient induced by a step change in wall temperature is the absence of infinite and very large heat fluxes, which are characteristic of the transient induced by a step in the wall temperature.

INTRODUCTION

This report analytically predicts the transient surface heat flux and temperature, on a cooled turbine blade or vane, caused by an abrupt change in the inlet temperature of the gas stream.

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In the design of blades and vanes for gas turbine engines, situations often arise where the fluid flowing over a solid boundary undergoes a transient, thus inducing time-dependent surface temperatures and heat flux. More often than not, a quasi-steady analysis is applied to such a problem; that is, it is assumed that the steady-state relations for the heat flux and the surface coefficient of heat transfer are approximately valid at each instant of time so long as the instantaneous values of temperature and velocity are inserted into these relations. That the quasi-steady approximation can induce appreciable error has been demonstrated in many publications, among them references 1 to 6. Sparrow (ref. 1) solves the problem of a step jump in the surface temperature at a two-dimensional planar stagnation point for laminar, constant-property flow when the velocity field is steady. By inserting Kantorovich velocity and temperature profiles into the integral form of the energy equation, he obtained approximate solutions for the transient surface heat flux and surface coefficients of heat transfer. In particular, he found that for small times, the process was that of pure conduction, with the result that the heat flux at the surface is inversely proportional to the square root of the time from the step change in temperature. Hence, very large (relative to steady-state values) surface heat fluxes are present during the initial portion of this type of transient. The linearity of the constant-property energy equation allows the response to the step change in wall temperature to be generalized to a wall temperature which varies arbitrarily with time. This he accomplished by way of the superposition integral. Siegel and Sparrow (ref. 2) solve for the transient surface heat flux due to a step change in wall temperature for the laminar thermal entrance region of flat ducts. Again, Kantorovich approximating sequences are used for the velocity profile (steady) and for the unsteady temperature profile. This time, however, insertion of these profiles into the integral form of the energy equation yields a partial differential equation for the thermal boundary layer thickness as a function of axial distance and time. They use the method of characteristics to arrive at the solution, which is then generalized by superposition to handle wall temperature variations that are arbitrary functions of time but are spatially uniform. The step change in surface flux and its generalization is also considered. The problem of transient heat transfer for laminar slug flow in ducts yields to the technique of separation of variables in an exact analytical solution by Siegel in reference 3. Cess (ref. 4) solves for the transient surface heat flux in laminar, constant-property, steady-velocity flow over a flat plate which has its surface temperature changed abruptly. He finds a series solution for small time in the physical plane, and one for long time in a Laplace transform plane. These limiting solutions are then joined, in an approximate manner, in the Laplace transform plane, and the result is inverted to give the surface flux over the entire time domain. Goodman (ref. 5) uses a Lighthill-type velocity profile in the integral energy equation to yield an equation which can be solved by characteristics for arbitrary free-stream velocity variation with position along the sur-

face of interest when the wall temperature abruptly changes. In reference 6, Chao and Jeng deal with unsteady heat transfer due to either wall temperature variation or wall flux variation with time at either a two-dimensional planar or an axisymmetric stagnation point. Sparrow and Siegel (ref. 7) consider unsteady turbulent heat transfer in tubes due to a step change in the wall temperature at time zero. Chambré (ref. 8) uses a double Laplace transformation to arrive at an exact analytical solution for the transient heat transfer in laminar slug flow of a fluid over a plate insulated on the bottom but containing an exponential generation term. Axial conduction in the plate is neglected and the temperature distribution in the transverse direction is lumped. Thus, the plate has no internal resistance to energy flow in that direction, but it does have finite thermal storage capacity. This finite thermal storage capacity couples the transient temperature distribution in the fluid to that of the plate and, hence, it becomes a transient conjugate problem (see Perelman, ref. 9). Transient local surface temperatures of the plate are presented for the case where both the plate and the fluid are at zero non-dimensional temperature initially, then the exponential generation term is turned on while the fluid approaching the leading edge of the plate stays at a nondimensional temperature of zero. Note is taken of the plane "wave" phenomenon peculiar to slug flow, namely that the "front" of fluid which is located at the leading edge, at the instant when the generation in the plate is turned on, separates the thermal fluid field into two vastly different regions. Ahead of this front, the process is one of conduction in the fluid with no mixing; whereas, behind the front there is mixing due to fluid being processed over the leading edge. Siegel (ref. 3) also discusses this effect in some detail. In reference 10, Soliman and Chambré use double Laplace transformations to find an exact analytical solution for the case of laminar, constant-property flow with a steady velocity distribution which is a linear function of distance away from a plate which undergoes either a step change in the surface temperature or in the heat flux.

In reference 11, Inouye and Yoshinaga use an approximate integral method to solve two problems of unsteady heat transfer at a stagnation point. In their first problem, the wall temperature is constant and the transient is caused by a step change in the fluid temperature. They indicate that there is no published work on this type of problem. However, it seems to this author that, near the stagnation point, there should not be any difference between the solution for a step change in wall temperature and their solution for a step change in the fluid temperature. This is so, because when the fluid temperature abruptly changes, it changes for all the fluid. There is no waiting for free-stream fluid which has had its temperature changed to be carried to any point on the body. Hence, the physical situation looks just like a fluid with constant temperature in contact with a wall, at a stagnation point, which has just had its temperature abruptly raised. Inspection of the relevant partial differential equations and the boundary and initial conditions also bears out this conclusion. As a final check, this author compared, for a

Prandtl number of unity, a few selected predictions for the surface heat flux from the paper of Inouye and Yoshinaga (ref. 11) with those of an earlier paper by Chao and Jeng (ref. 6) and found excellent agreement between them. In the second stagnation-point problem worked in reference 11, the wall has its rear surface insulated and is initially at the temperature of the fluid when suddenly the fluid temperature is changed. Axial conduction was neglected in the plate and the plate temperature was lumped in the transverse direction, but the plate had finite thermal storage capacity. Hence, this is a transient conjugate problem, as was the problem in reference 8. Approximate results for the surface heat flux and wall temperature are given as a function of time for various values of a transient coupling parameter. Lyman (ref. 12) solves for the wall heat flux and wall temperature at a stagnation point when the free stream suddenly experiences a step change in its temperature. His is a conjugate problem which he solves in an approximate fashion using Kantorovich profiles in the integral form of the energy equation for the fluid while retaining the partial differential equation for the solid wall. For the case of a semi-infinite wall an approximate solution for all values of time is found. Short- and long-time approximate solutions are found for a finite thickness plate insulated on its lower surface. Unlike the approach of reference 11, Lyman applies the step change in free-stream temperature to the outer edge of a thermal boundary layer and calculates the transit time needed for the wall to feel the effects of this change. Thus, for times less than the transit time, the wall flux is zero. The flux is never infinite even at the time when the wall first learns of the free-stream temperature change. By that time a smooth continuous temperature profile has developed in the thermal boundary layer, thus negating the possibility of an infinite temperature gradient at the wall.

Except for the conjugate problems of references 8, 11, and 12 and the present report, there does not seem to be any literature on transient conjugate problems. However, some steady-state conjugate problems in forced-convection heat transfer can be found in references 9 and 13 to 17.

The instigation for the present work was the transient in the fluid, flowing over gas turbine blades or vanes, induced by a change in the fluid inlet temperature due to startup or due to a change in the power level of an already operating engine. One wishes to predict the transient surface flux and the surface temperature variation. This problem, unlike the other transient heat-transfer problems already solved in the references, is unique in the sense that the reason for the transient is not some prescribed wall temperature variation with time. Rather the transient occurs because of the change in the fluid temperature at the leading edge. In addition, a blade or vane does not usually have some prescribed temperature but rather has prescribed internal coolant conditions. And hence, the solid over which the fluid flows has its temperature distribution coupled to that of the fluid - a conjugate problem.

ANALYSIS

A cooled gas turbine blade or vane will be idealized as a thin flat plate of thickness b which has its lower surface exposed to coolant at a constant temperature T_c and which has a constant surface coefficient of heat transfer h_c between the coolant and the lower surface. The hot combustor gases at temperature T_∞ and velocity u_∞ flow over the top of the plate as depicted in figure 1. (All symbols are defined in appendix A.)

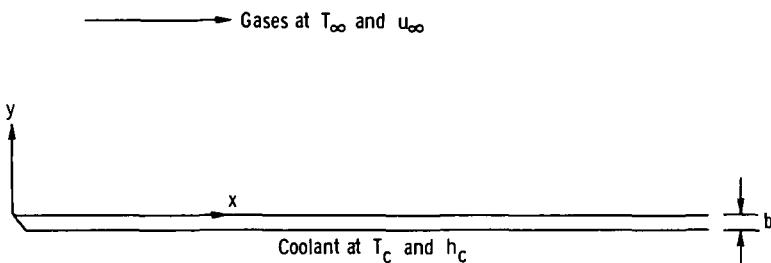


Figure 1. - Schematic diagram of physical situation.

The analysis is for laminar, constant-property, two-dimensional planar, boundary-layer-type flow without appreciable viscous dissipation.

Solution for Step Change in Fluid Inlet Temperature When Fluid and Plate Are at Coolant Temperature

At time $t = 0$ both the plate and the fluid flowing over the plate are at the coolant temperature T_c when suddenly the fluid for $x \leq 0$ has its temperature changed to a value T_∞ different from T_c . (The problem of a step change in the inlet temperature after the steady state has been reached, which occurs when going to a different power level or in shutdown of the engine, can be easily handled once the solution to the case just described is available.) The surface-heat-flux, plate-surface-temperature, and temperature distributions within the moving fluid are required for all values of time $t > 0$. In order to investigate the general character and trends of such a solution, some further idealizations are made which permit an exact analytical solution. These idealizations are a steady slug flow velocity field (which corresponds to vanishingly small Prandtl numbers), neglect of axial conduction in the plate (which can sometimes be justified physically on the basis that cooled blades and vanes are relatively thin walled and constructed of relatively low-thermal-conductivity material), and lumping of the temperature distribution in the y -direction in the plate (which can often be justified if

the plate Biot number, based upon b and the larger of the coolant-side surface coefficient and the gas-side surface coefficient, is relatively small). With $\theta = T - T_\infty$, the partial differential form of the thermal energy equation for the fluid is (see, e.g., ref. 18)

$$\frac{\partial \theta}{\partial t} + u_\infty \frac{\partial \theta}{\partial x} = \alpha_f \frac{\partial^2 \theta}{\partial y^2} \quad (1)$$

The boundary conditions are

$$t = 0 \quad x > 0 \quad y > 0 \quad \theta = \theta_c = T_c - T_\infty \quad (2)$$

$$x = 0 \quad t > 0 \quad y > 0 \quad \theta = 0 \quad (3)$$

$$y \rightarrow \infty \quad t > 0 \quad x > 0 \quad \theta \text{ is finite} \quad (4)$$

Next make an energy balance on the plate material, neglecting axial conduction and thermal resistance in the y -direction. Note that the plate temperature at any x and t must equal the fluid temperature at the same value of x and t . Thus, the partial differential equation for the plate temperature becomes a boundary condition for equation (1), namely,

$$y = 0 \quad t > 0 \quad x > 0 \quad \frac{\partial \theta}{\partial y} = \frac{h_c}{k_f} (\theta - \theta_c) + r \frac{\partial \theta}{\partial t} \quad (5)$$

where $r = \rho_s C_{p,s} b / k_f$ and is a measure of the thermal storage capacity, per unit length, of the plate material.

The solution of equation (1), subject to the side conditions (2), (3), (4), and (5), was approached by way of two successive Laplace transformations. First, the Laplace transformations of the equation and the appropriate boundary conditions with respect to time were taken. The transformed temperature function in the first transformed plane is

$$\bar{\theta} = \mathcal{L}[\theta] = \int_0^\infty \theta e^{-pt} dt \quad (6)$$

Performing the indicated transformation yields

$$p\bar{\theta} - \theta_c + u_\infty \frac{\partial \bar{\theta}}{\partial x} = \alpha_f \frac{\partial^2 \bar{\theta}}{\partial y^2} \quad (7)$$

$$x = 0 \quad y > 0 \quad \bar{\theta} = 0$$

$$y \rightarrow \infty \quad x > 0 \quad \bar{\theta} \text{ is finite} \quad (8)$$

$$y = 0 \quad x > 0 \quad \frac{\partial \bar{\theta}}{\partial y} = \frac{h_c}{k_f} \left(\frac{\bar{\theta}}{s} - \frac{\theta_c}{p} \right) + r(p\bar{\theta} - \theta_c)$$

Next, a Laplace transformation with respect to x is taken for equation (7) and the last two equations of the set (8). The transformed temperature function in the second transformed plane is defined as

$$\bar{\bar{\theta}} = \mathcal{L}[\bar{\theta}] = \int_0^\infty \bar{\theta} e^{-sx} dx \quad (9)$$

This operation yields the following ordinary differential equation and its associated boundary conditions:

$$\frac{d^2 \bar{\bar{\theta}}}{dy^2} - \left(\frac{u_\infty s + p}{\alpha_f} \right) \bar{\bar{\theta}} = - \frac{\theta_c}{s \alpha_f} \quad (10)$$

$$y \rightarrow \infty \quad \bar{\bar{\theta}} \text{ is finite} \quad (11)$$

$$y = 0 \quad \frac{d \bar{\bar{\theta}}}{dy} = \left(\frac{h_c}{k_f} + rp \right) \bar{\bar{\theta}} - \theta_c \left(\frac{r}{s} + \frac{h_c}{k_f s p} \right) \quad (12)$$

Solving equation (10) and subjecting its solution to condition (11) and to condition (12), which is actually the conjugation condition, yield the solution for $\bar{\bar{\theta}}$ as

$$\bar{\bar{\theta}} = \frac{u_\infty \theta_c \left(\frac{h_c}{k_f} + rp \right) e^{-y \sqrt{(u_\infty s + p)/\alpha_f}}}{p(u_\infty s + p) \left(\sqrt{\frac{u_\infty s + p}{\alpha_f}} + \frac{h_c}{k_f} + rp \right)} + \frac{\theta_c}{s(u_\infty s + p)} \quad (13)$$

Since one is primarily interested in the surface-temperature distribution and the surface-heat-flux distribution, he can, in the spirit of Lighthill (ref. 19) invert equation (13) with $y = 0$ and invert $d\bar{\theta}/dy$ at $y = 0$, rather than the entire temperature distribution. Since the successive inversion with respect to x and t of equation (13) is not all that formidable, the author has inverted equation (13) and then computed the wall temperature and surface heat flux by operations in the x, y, t -plane. As an internal check the transformed surface-heat-flux and surface-temperature distribution in the s, y, p -plane was also inverted and exact agreement was noted. All the inversions were performed by using tables of transforms (specifically, refs. 20 and 21) in conjunction with the substitution, translation, and convolution theorems of the operational mathematics. Details are given in appendix B.

When the nondimensional variables

$$\tau = \frac{u_{\infty} t}{x}$$

$$Y = \frac{y}{2} \sqrt{\frac{u_{\infty}}{\alpha_f x}}$$

$$\eta = \frac{h_c}{k_f} \sqrt{\frac{\alpha_f x}{u_{\infty}}}$$

and

$$\epsilon = \frac{\sqrt{\frac{x}{\alpha_f u_{\infty}}}}{2r}$$

are introduced, the exact analytical solution for the nondimensional temperature excess ratio in the moving fluid becomes

$$\frac{\theta}{\theta_c} = 1 + u(\tau - 1) \left(-1 + \operatorname{erfc}[Y] - e^{2\eta Y + \eta^2} \{ \operatorname{erf}[\epsilon(\tau - 1) + \eta + Y] - \operatorname{erf}[\eta + Y] \} \right) \quad (14)$$

where $u(\tau - 1)$ is the unit step function, that is,

$$u(\tau - 1) \begin{cases} = 0 & \text{for } \tau < 1 \\ = +1 & \text{for } \tau > 1 \end{cases}$$

The surface-temperature variation is found by setting $Y = 0$ in equation (14), which gives

$$\frac{\theta_w}{\theta_c} = 1 + u(\tau - 1) \left(-e^{\eta^2} \{ \text{erf}[\epsilon(\tau - 1) + \eta] - \text{erf}[\eta] \} \right) \quad (15)$$

The nondimensional surface heat flux Q_w is defined as follows:

$$Q_w = \frac{q_w}{k_f \theta_c \sqrt{\frac{u_\infty}{\pi \alpha_f x}}} \quad (16)$$

The denominator in equation (16) is the surface heat flux that would obtain in steady, slug flow over an isothermal plate at temperature excess equal to θ_c .

It can be readily shown that

$$Q_w = - \frac{\sqrt{\pi}}{2\theta_c} \left(\frac{\partial \theta}{\partial Y} \right)_{Y=0}$$

With this and equation (14), the exact analytical solution for the nondimensional surface heat flux becomes

$$Q_w = \left. \begin{cases} e^{-[\epsilon^2(\tau-1)^2 + 2\eta\epsilon(\tau-1)]} + \sqrt{\pi} \eta e^{\eta^2} \{ \text{erf}[\epsilon(\tau - 1) + \eta] - \text{erf}[\eta] \} & \text{for } \tau \geq 1 \\ 0 & \text{for } \tau < 1 \end{cases} \right\} \quad (17)$$

The exact analytical solution for the nondimensional temperature is seen to satisfy the side conditions of equation (1) and, as a final check on the solution technique, equation (14) was substituted into the governing partial differential equation (1) and was found to satisfy it identically in the domain of definition of θ . Some limiting cases of equations (15) and (17) have also been checked to ensure that they agree with physical reasoning. If $\eta \rightarrow \infty$, interpreted as $h_c \rightarrow \infty$, the plate must be at the coolant temperature excess θ_c , and the nondimensional flux Q_w should reduce to that appropriate to an

isothermal flat plate, namely unity. By applying L'Hopital's rule to the second term of equations (15) and (17), it is readily seen that, as

$$\eta \rightarrow \infty \quad \theta_w \rightarrow \theta_c \quad Q_w \rightarrow 1 \quad (18)$$

The coupling parameter $\epsilon \rightarrow 0$ corresponds to infinite thermal storage capacity of the plate. And since the plate is initially at θ_c , it would behave, with $\epsilon \rightarrow 0$, as a heat reservoir at θ_c for all values of time. Hence, Q_w should be unity. Equations (15) and (17) properly reduce to this degenerate case also.

Solution for Step Change in Fluid Inlet Temperature

After Steady State Had Been Reached

The eventual steady state reached in the previous case can be found by letting the nondimensional time $\tau \rightarrow \infty$ in equations (14), (15), and (17). This yields

$$\frac{\theta_{ss}}{\theta_c} = \operatorname{erfc}[Y] - e^{2\eta Y + \eta^2} \operatorname{erfc}[\eta + Y] \quad (19)$$

$$\frac{\theta_{w,ss}}{\theta_c} = 1 - e^{\eta^2} \operatorname{erfc}[\eta] \quad (20)$$

$$Q_{w,ss} = \sqrt{\pi} \eta e^{\eta^2} \operatorname{erfc}[\eta] \quad (21)$$

where $Q_{w,ss}$ denotes the eventual steady-state value of Q_w and similarly for the functions θ_{ss} and $\theta_{w,ss}$.

Now the following problem is considered: the plate of figure 1 is in the steady state described by equations (19), (20), and (21), where the fluid inlet temperature is T_∞ , when suddenly the inlet temperature of the fluid is changed to a different value, T_0 , which corresponds to an increase or a decrease of the power level of a gas turbine engine (a so-called acceleration or deceleration). Once again, we would like to predict the temperature distribution within the moving fluid and the surface-heat-flux and surface-temperature distributions in this type of transient. The governing partial differential equation is still equation (1) and side conditions (4) and (5) remain the same. Side conditions (2) and (3), however, take the following form:

$$t = 0 \quad x > 0 \quad y > 0 \quad \theta = \theta_{ss} \quad (22)$$

where θ_{ss} is the function of x and y given in equation (19), and

$$x = 0 \quad t > 0 \quad y > 0 \quad \theta = T_0 - T_\infty \quad (23)$$

Because of the case previously solved, it is natural to seek a solution to this case by using the following decomposition:

$$\theta(x, y, t) = \varphi(x, y, t) + \theta_{ss}(x, y) \quad (24)$$

Inserting equation (24) into equations (1), (4), (5), (22), and (23) yields

$$\varphi(x, y, t) = T_0 - T_\infty + \gamma(x, y, t) \quad (25)$$

where $\gamma(x, y, t)$ satisfies equations (1) to (5) if θ_c is replaced by $T_\infty - T_0$. Thus, the solution for $\gamma(x, y, t)$ is equation (14) with θ_c replaced by $T_\infty - T_0$ and, of course, γ in place of θ . Doing this and using equations (25), (24), and (19) yields the exact analytical solution for the temperature distribution within the moving fluid for this type of transient as follows (details in appendix C):

$$\begin{aligned} \frac{\theta}{\theta_c} &= \operatorname{erfc}[Y] - e^{2\eta Y + \eta^2} \operatorname{erfc}[\eta + Y] \\ &+ \left[\frac{T_0 - T_\infty}{T_\infty - T_c} \right] u(\tau - 1) \left(-1 + \operatorname{erfc}[Y] - e^{2\eta Y + \eta^2} \{ \operatorname{erf}[\epsilon(\tau - 1) + \eta + Y] - \operatorname{erf}[\eta + Y] \} \right) \end{aligned} \quad (26)$$

Setting $Y = 0$ in equation (26) gives the surface temperature excess ratio as

$$\frac{\theta_w}{\theta_c} = 1 - e^{\eta^2} \operatorname{erfc}[\eta] - u(\tau - 1) \left(\frac{T_0 - T_\infty}{T_\infty - T_c} \right) e^{\eta^2} \{ \operatorname{erf}[\epsilon(\tau - 1) + \eta] - \operatorname{erf}[\eta] \} \quad (27)$$

By operating on equation (26) and using equation (21), the transient surface heat flux Q'_w can be put in the following convenient form:

$$\frac{Q'_w}{Q_{w,ss}} = 1 + \left(\frac{T_0 - T_\infty}{T_\infty - T_c} \right) \left(\frac{Q_w}{Q_{w,ss}} \right) \quad \text{for } \tau > 1$$

$$\frac{Q'_w}{Q_{w,ss}} = 1 \quad \text{for } \tau < 1 \quad (28)$$

For $\tau > 1$, $Q_{w,ss}$ is given by equation (21) while Q_w is given by equation (17).

RESULTS AND DISCUSSION

The first case considered is where both the fluid and the plate are initially at temperature T_c and suddenly the temperature of the inlet fluid at $x = 0$ is changed to T_∞ and held there. Plots are presented, for this case, of nondimensional surface heat flux Q_w and nondimensional temperature excess ratio θ_w/θ_c in figures 2 and 3, respectively, as functions of a time parameter $\epsilon(\tau - 1)$. The plots are presented for various values of η ranging from $\eta = 0$ (lower plate surface insulated) to $\eta \rightarrow \infty$ (which corresponds to a constant plate temperature T_c). The values of η between these two extremes were chosen not only to illustrate the trends of the flux and the temperature with

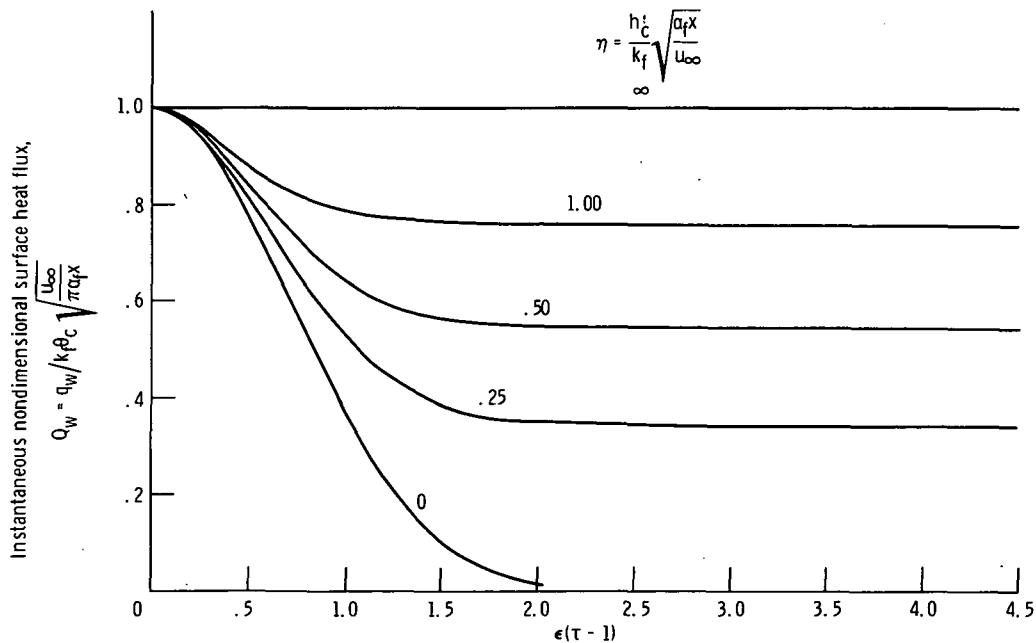


Figure 2. - Heat-transfer response to step change in inlet temperature when fluid and plate are initially at coolant temperature.

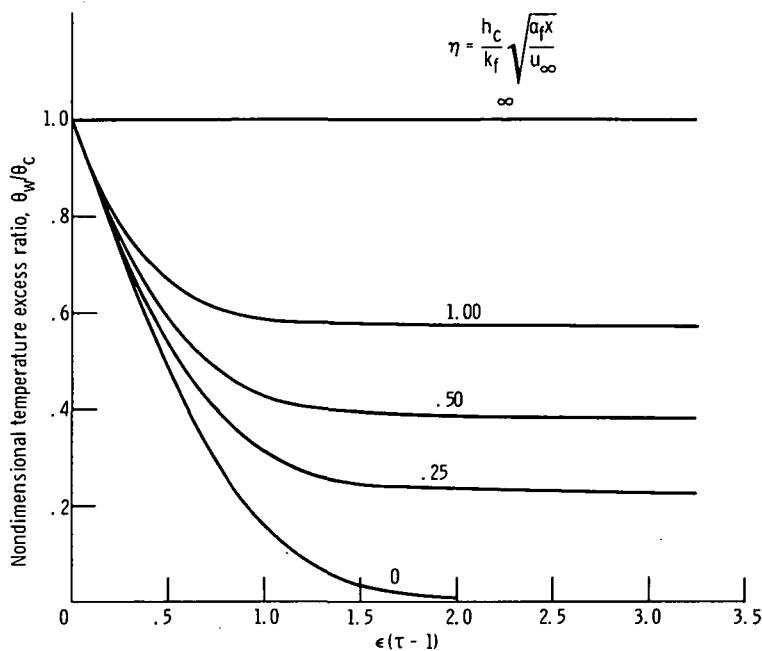


Figure 3. - Wall temperature response to step change in inlet temperature when fluid and plate are initially at coolant temperature.

η , but also to be in a reasonable range of possible application. A study of η shows, by its definition, that it is a measure of the coolant-side surface coefficient divided by the local gas-side surface coefficient. In the range of gas turbine application, this parameter could vary from about 0 to unity, with $\eta = 0$ corresponding to a solid uncooled blade or vane and with $\eta = 1$ corresponding to a relatively sophisticated internal cooling scheme such as impingement or impingement with crossflow. The choice of $\epsilon(\tau - 1)$ as the abscissa rather than $\epsilon\tau$ is dictated by the fact that because of the unit step function in the solution, Q_w is zero for $\tau < 1$ and only attains nonzero values for $\tau > 1$. A similar statement holds for θ_w/θ_c . This is a result of the fact that the plate at position x does not realize the inlet temperature has changed until the fluid that was at the inlet at $\tau = 0$ reaches the position of interest. Inspection of figure 2 shows that as η decreases there is a more pronounced variation of flux Q_w with time, as would be expected based on the previous discussion of the limiting case $\eta \rightarrow \infty$, in which there is really no transient in the plate. Study of figures 2 and 3 allows determination of the value that $\epsilon(\tau - 1)$ must be greater than or equal to so that both the surface flux and the surface temperature excess ratio are within 5 percent of their steady-state values. This critical value of $\epsilon(\tau - 1)$ is denoted as a^* ; the values of a^* are presented in the following table for various values of η :

$\eta = \frac{h_c}{k_f} \sqrt{\frac{\alpha_f x}{u_\infty}}$	Value which a time parameter, $\epsilon(\tau - 1)$, must be greater than for solution to be within 5 percent of steady state, a^*
0.25	1.75
.50	1.375
1.00	.875
∞	0

For $\eta = 0$, this type of criterion does not hold since the steady-state values of Q_w and θ_w/θ_c are both zero. For this case, $\eta = 0$, it may be noticed that for $\epsilon(\tau - 1) \geq 2$, both Q_w and θ_w/θ_c are less than 0.02.

The influence of the coupling parameter ϵ can be seen by reference to figure 4 where Q_w is plotted versus $\tau - 1$ for different values of ϵ at two different values of η . One immediate observation is that the eventual steady-state Q_w is independent of ϵ and dependent only on η . That this should be so can be verified by recalling that neglect of axial conduction in the plate and the lumping of the plate temperature in the y -direction causes the plate temperature to be coupled to that of the fluid only through the thermal storage term of the plate. And in the steady state, the thermal storage capacity of the plate is immaterial since the plate temperature does not change with time. Side condition (5) shows this also since in the steady state $\partial\theta/\partial t = 0$, and the solution must be

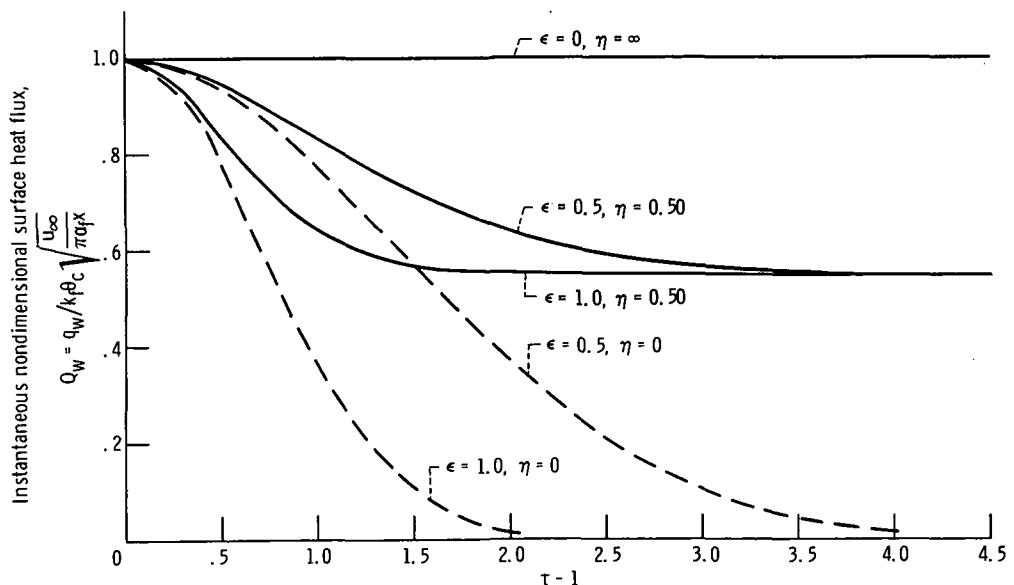


Figure 4. - Influence of coupling parameter ϵ on heat-transfer response to step change in inlet temperature.

independent of r and therefore of ϵ . Also the effect of increasing ϵ on the nondimensional flux Q_w is decreased with increasing η , as would be expected from the prior discussion of the limiting case $\eta \rightarrow \infty$.

A physical interpretation of the coupling parameter ϵ can be made as follows: From its definition and the definition of r ,

$$\epsilon = \frac{\sqrt{\frac{x}{\alpha_f u_\infty}}}{\frac{2\rho_s C_{p,s}^b}{k_f}}$$

After some algebraic manipulation this becomes

$$\epsilon = \frac{\rho_f C_{p,f} \sqrt{\frac{\alpha_f x}{u_\infty}}}{2\rho_s C_{p,s}^b} \quad (29)$$

But for steady, laminar, slug flow over an isothermal plate, the thermal boundary layer thickness is proportional to the square root appearing in the numerator of equation (29). That is,

$$\delta_t \sim \sqrt{\frac{\alpha_f x}{u_\infty}}$$

Therefore,

$$\epsilon \sim \frac{\rho_f C_{p,f} \delta_t}{\rho_s C_{p,s}^b}$$

Thus, ϵ is a measure of the ratio of the thermal storage capacity of the boundary-layer fluid per unit length to the thermal storage capacity of the solid plate per unit length.

Hence, when ϵ is very small, $\epsilon \approx 0$, the plate temperature excess will remain at θ_c , giving the previously discussed limiting case of $Q_w = 1$. At the other extreme, ϵ ap-

proaches ∞ , meaning that the plate has essentially zero thermal storage capacity compared to the fluid and thus reacts to impressed thermal loadings instantly and without hindering the fluid. This case causes a rather degenerate transient in which Q_w is zero for $\tau < 1$ and equal to its steady-state value for all $\tau > 1$, the steady-state value being the function of η given by allowing ϵ to approach ∞ in equation (17). Doing this gives

$$Q_w \rightarrow \sqrt{\pi} \eta e^{\eta^2} \operatorname{erfc}[\eta] \quad \text{as } \epsilon \rightarrow \infty \quad \text{for } \tau > 1$$

For this case, once τ is greater than unity, the deviation of Q_w from unity results solely from the thermal history effect. (Recall that the definition of Q_w requires that it be unity for an isothermal surface at temperature excess θ_c .) An interesting observation from figures 2 and 4 is that the nondimensional flux never achieves the infinite or extremely large values which are characteristic of solutions in which there is an abrupt step in the wall temperature caused by the pure conduction process ahead of the fluid front located at $x = 0$ at $t = 0$. This observation agrees with the results presented in reference 12, where a transient stagnation-point problem was worked for a step in free-stream temperature which required a nonzero transit time to be felt at the solid surface.

When trying to establish conclusions based on figures 2 or 4, we must keep in mind that the surface flux has been normalized by dividing it by the steady-state surface flux for an isothermal flat plate at θ_c . If we compare the ratio of the instantaneous flux for

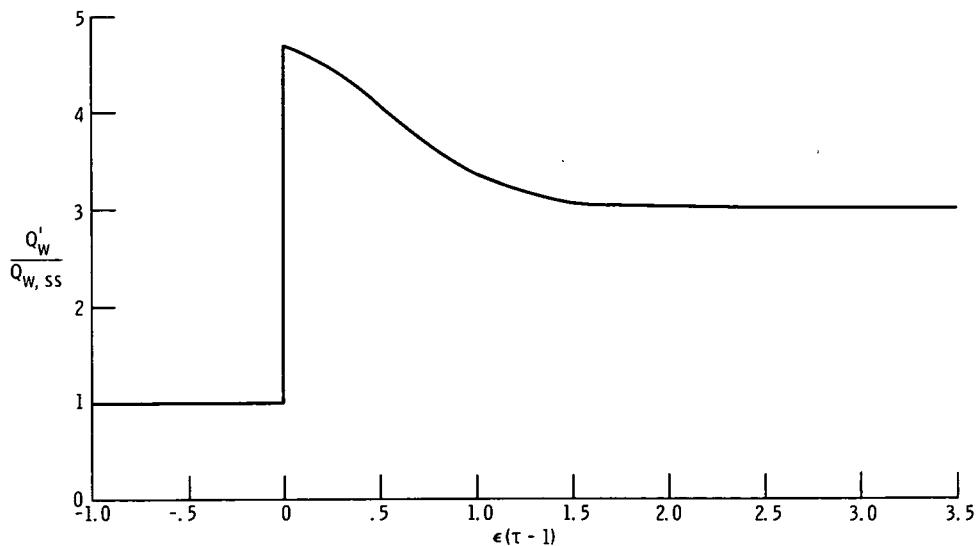


Figure 5. - Heat-transfer response to step change in inlet temperature when fluid and plate are in an initial steady state with their temperature not equal to coolant temperature. $\eta = 0.50$; $(T_0 - T_\infty)/(T_\infty - T_c) = 2.0$.

particular values of η , ϵ , and τ to the eventual steady-state flux at the same η and ϵ , and the picture, of course, is vastly different. For instance, at $\eta = 0.50$ and $\tau = 1$, the ratio of the surface flux at $\tau = 1$ to the eventual steady-state flux ($\tau \rightarrow \infty$) is 1.833.

In figure 5, one representative curve for the transient flux is given for the situation where steady-state conditions originally prevailed. Thus, the surface temperature and flux are given by equations (20) and (21), respectively, and the inlet temperature is T_∞ , when suddenly the inlet temperature is increased to the value T_0 . For $\eta = 0.50$ and $(T_0 - T_\infty)/(T_\infty - T_c) = 2$, the ratio of the instantaneous nondimensional flux Q'_w to the flux $Q_{w,ss}$ which existed in the original steady state is plotted versus the time parameter. As shown in figure 5, $Q'_w = Q_{w,ss}$ for all $\tau < 1$. Then, when the fluid that was at $x = 0$ when the inlet temperature was changed arrives at any position corresponding to $\tau = 1$, the flux increases and then gradually decays to its new higher steady-state value.

The simplest approach to transient convection heat transfer is the quasi-steady analysis where we assume that the steady-state relations for the wall heat flux are approximately valid at each instant of time if the wall temperature at each instant of time is used. For the laminar, constant-property, slug-flow problem, the quasi-steady wall flux, $q_{w,qs}$ is given by

$$q_{w,qs} = k_f \sqrt{\frac{u_\infty}{\pi \alpha_f}} \int_0^x \frac{1}{\sqrt{x-\xi}} \frac{\partial \theta_w}{\partial \xi} d\xi \quad (30)$$

Equation (30) is derived in appendix D by using Lighthill's approach (ref. 19).

An energy balance on a length dx of plate using equation (30) yields the following equation for the quasi-steady wall temperature excess:

$$\frac{\partial \theta_w}{\partial t} + a_o \int_0^x \frac{1}{\sqrt{x-\xi}} \frac{\partial \theta_w}{\partial \xi} d\xi = b_o (\theta_c - \theta_w) \quad (31)$$

where

$$a_o = \frac{1}{r} \sqrt{\frac{u_\infty}{\pi \alpha_f}}$$

and

$$b_0 = \frac{h_c}{\rho_s C_p, s b}$$

Two successive Laplace transformations, for the cases where $\theta_w = \theta_c$ at $t = 0$ and $\theta_w = 0$ at $x = 0$, reduce equation (31) to an algebraic equation that can be inverted to give the following quasi-steady wall temperature distribution:

$$\frac{\theta_w, qs}{\theta_c} = 1 - e^{\eta^2} \{ \operatorname{erf}[\epsilon\tau + \eta] - \operatorname{erf}[\eta] \} \quad (32)$$

The quasi-steady flux is most easily calculated from equation (31) since the integral term is directly related to the flux through equation (30) and θ_w is now known from equation (32). The result for the nondimensional quasi-steady flux $Q_{w, qs}$ is

$$Q_{w, qs} = e^{-(\epsilon^2 \tau^2 + 2\eta\epsilon\tau)} + \sqrt{\pi} \eta e^{\eta^2} \{ \operatorname{erf}[\epsilon\tau + \eta] - \operatorname{erf}[\eta] \} \quad (33)$$

Figure 6 shows, for $\eta = 0.5$ and $\epsilon = 1$, the quasi-steady flux given by equation (33) as the dashed line and the exact transient flux given by equation (17) as a solid line. As shown, the quasi-steady approximation is not very good for this choice of parameters η and ϵ . In particular, the quasi-steady solution does not predict the lag time $\tau = 1$ needed before a position on the plate can know that the inlet fluid temperature has changed. Lyman, reference 12, notes this also in connection with his transit time. In

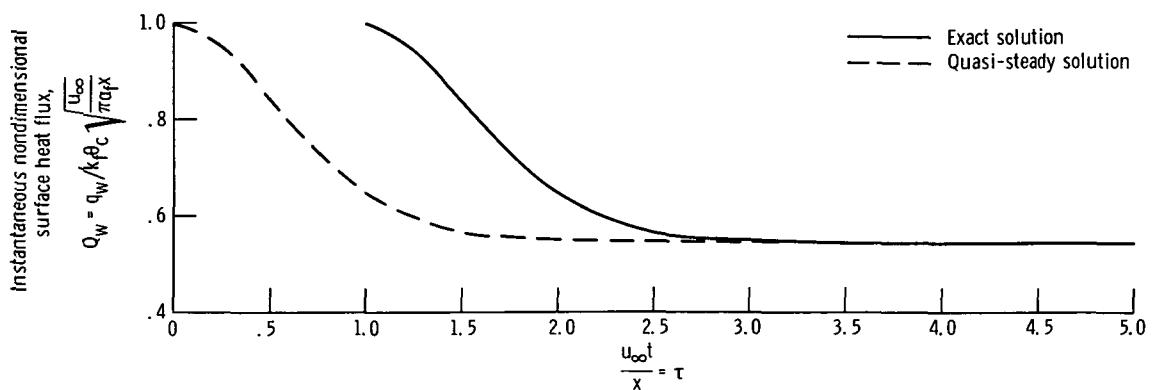


Figure 6. - Comparison of exact and quasi-steady solutions for heat-flux response to step change in inlet temperature when both fluid and plate were originally at coolant temperature. $\epsilon = 1$; $\eta = 0.5$.

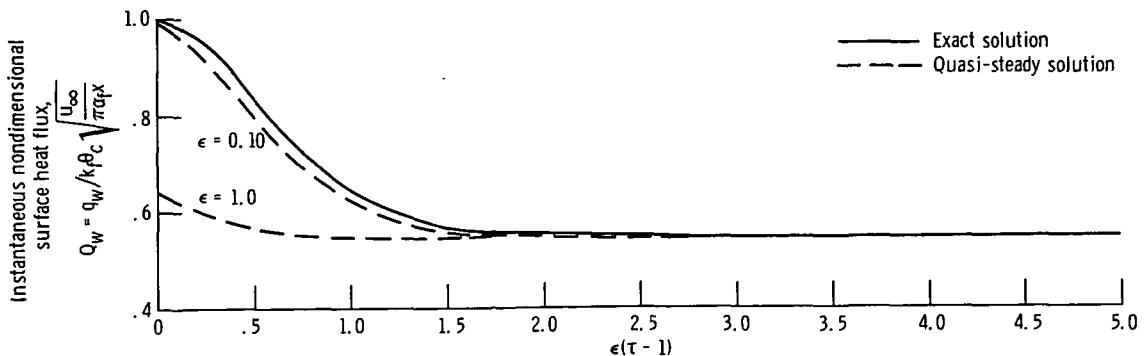


Figure 7. - Influence of coupling parameter ϵ on quasi-steady solution for heat-flux response to step change in inlet temperature when both fluid and plate were originally at coolant temperature. $\eta = 0.5$.

figure 7 the nondimensional wall flux is plotted versus $\epsilon(\tau - 1)$ at $\eta = 0.5$. (The part of the quasi-steady solution for $\tau < 1$ has been deleted.) Again the solid curve is the exact solution (eq. (17)), while the quasi-steady approximation (eq. (33)) is shown as a dashed curve for two values of ϵ . As would be expected, the quasi-steady solution approaches the exact solution for small values of the coupling parameter ϵ . A comparison between the quasi-steady solutions (eqs. (32) and (33)) and the corresponding exact solutions for the same case (eqs. (15) and (17)) yields the conclusion that the quasi-steady values for both temperature excess ratio and wall flux will be in error by less than 10 percent relative to the exact solutions, over the time range $\tau \geq 1$ if $\epsilon \leq 0.08$ for $\eta = 0.5$ and $\epsilon \leq 0.20$ for $\eta = 1$. There is inherent difficulty in making this comparison for the $\eta = 0$ case since the exact solutions tend to zero with increasing time. However, if $\epsilon \leq 0.02$, the error in the temperature excess ratio and the flux will be less than 10 percent for $\eta = 0$ if $0 \leq \epsilon(\tau - 1) \leq 1.8$.

It is interesting to note that if τ in the quasi-steady solution is just replaced by $\tau - 1$, the quasi-steady solution agrees with the exact solution. Quasi-steady solution behavior of this general type was also noted by Lyman (ref. 12) in connection with the stagnation-point solution. On this basis it would seem reasonable, as a heuristic attempt to getting a better approximation, to just replace τ by $\tau - 1$ in quasi-steady solutions to the more difficult transient convection problems involving a nonslug velocity profile. However, this procedure would not be used if axial conduction in the plate were significant, because downstream portions of the plate and the fluid would not have to wait until the fluid front that was at $x = 0$ at $t = 0$ reached them in order that they respond to a change in inlet conditions.

Finally, a representative calculation of ϵ was made for a turbine vane to see whether a simpler quasi-steady approach is valid. The conditions used were $T_\infty = 1390$ K (2500^0 R), a pressure of 82.6 N/cm 2 (120 psia), $u_\infty = 366$ m/sec (1200 ft/sec), $b = 0.102$ cm (0.040 in.), $x = 3.25$ cm (1.28 in.), and $\rho_s C_{p,s} \approx 3.4$ J/(cm 3)(K)

(50 Btu/(ft³)(R)). The fluid properties were those of air. This yielded $\epsilon = 1.94 \times 10^{-5}$. Hence, based on the idealized model analyzed herein, it would appear as if a quasi-steady analysis would be sufficient for gas turbine blades and vanes. Lyman's computations for the stagnation-point region, in reference 12, also seems to point to this conclusion. Further investigation, however, is needed to see how the inclusion of the effects of axial conduction and nonslug velocity profiles would modify this conclusion.

SUMMARY OF RESULTS

By using two successive Laplace transformations, exact analytical solutions for the transient gas temperature distribution and surface heat flux are found for the case where the gas, flowing over a cooled vane or blade, experiences a step change in its inlet temperature which then causes both the gas temperature and the vane or blade temperature to vary with time.

1. Equations and curves are presented from which the temperature distribution within the gas and the surface heat flux can be calculated.
2. Quasi-steady results are also derived and compared with the exact transient solution, yielding a criterion for use of quasi-steady results with less than 10 percent error.
3. The theoretical model predicts, in this type of transient, the absence of the infinite or very large surface heat fluxes associated with transients initiated by a step change in the wall temperature.
4. A criterion is derived which gives the time needed for the wall flux and the wall temperature to reach a value within 5 percent of their steady-state values.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 14, 1972,
501-24.

APPENDIX A

SYMBOLS

a^*	value which a time parameter, $\epsilon(\tau - 1)$, must be greater than for solution to be within 5 percent of steady state
a_0	$\sqrt{\frac{u_{\infty}}{\pi \alpha_f}} / r$
b	plate thickness
b_o	$h_c / \rho_s C_{p,s} b$
C_p	constant-pressure specific heat
c	$(h_c / rk_f) + (yu_{\infty} / 2r\alpha_f x)$
d	$r \sqrt{\frac{\alpha_f x}{u_{\infty}}}$
h_c	coolant-side surface coefficient of heat transfer
k_f	thermal conductivity of fluid flowing over upper surface of plate
p	Laplace transform parameter
Q_w	instantaneous nondimensional surface heat flux
Q'_w	same as Q_w but for transient initiated after steady state has been reached
$Q_{w,ss}$	eventual steady-state value of Q_w
$Q_{w,qs}$	quasi-steady equivalent of Q_w
q_w	instantaneous surface heat flux
$q_{w,qs}$	quasi-steady instantaneous surface heat flux
r	$\rho_s C_{p,s} b / k_f$
s	Laplace transform parameter
T	temperature
T_c	coolant temperature
T_0	an inlet temperature

T_∞	an inlet temperature
t	time
$u(\tau - 1)$	unit step function, equals 0 for $\tau < 1$, equals +1 for $\tau > 1$
u_∞	free-stream velocity
x	space coordinate along plate
Y	nondimensional y-coordinate
y	space coordinate perpendicular to plate
α_f	thermal diffusivity of fluid flowing over plate
β	expression defined by eq. (B12)
γ	temperature excess defined by eq. (B7)
δ_t	thermal boundary-layer thickness
ϵ	$\sqrt{\frac{x}{\alpha_f u_\infty}} / 2r$
θ	temperature excess, $T - T_\infty$
θ_c	coolant temperature excess
θ_{ss}	steady-state value of θ
θ_w	value of θ at wall
φ	temperature excess defined by eq. (24)
φ_0	$T_0 - T_\infty$
ρ	density
λ	dummy variable for time
ξ	dummy variable for x
σ	$(h_c/k_f) + rp$
η	$(h_c/k_f) \sqrt{\alpha_f x / u_\infty}$
τ	nondimensional time, $u_\infty t / x$

Subscripts:

c	coolant
f	properties of fluid flowing over plate

qs quasi-steady conditions
s plate properties
ss steady state
w wall conditions
w, ss wall conditions at eventual steady state
 ∞ free-stream conditions

Superscripts:

- function in first transformed plane
- =** function in second transformed plane

APPENDIX B

DETAILS OF THE INVERSION OF EQUATION (13) OF THE ANALYSIS SECTION

Equation (13) gives the temperature excess distribution in the second transformed plane as

$$\bar{\bar{\theta}} = \frac{u_{\infty} \theta_c \left(\frac{h_c}{k_f} + rp \right) e^{-y \sqrt{(u_{\infty}s + p)/\alpha_f}}}{p(u_{\infty}s + p) \left(\sqrt{\frac{u_{\infty}s + p}{\alpha_f}} + \frac{h_c}{k_f} + rp \right)} + \frac{\theta_c}{s(u_{\infty}s + p)} \quad (13)$$

After defining

$$\sigma = \frac{h_c}{k_f} + rp \quad (B1)$$

and rearranging, equation (13) becomes

$$\frac{\bar{\bar{\theta}}}{\theta_c} = \frac{\sqrt{\frac{\alpha_f}{u_{\infty}}} \sigma e^{-y \sqrt{u_{\infty}/\alpha_f} \sqrt{s + (p/u_{\infty})}}}{p \left(s + \frac{p}{u_{\infty}} \right) \left(\sqrt{s + \frac{p}{u_{\infty}}} + \sigma \sqrt{\frac{\alpha_f}{u_{\infty}}} \right)} + \frac{1}{u_{\infty}s \left(s + \frac{p}{u_{\infty}} \right)} \quad (B2)$$

Now the inverse Laplace transformation of equation (B2) with respect to x must be found. The second term is found to be

$$\mathcal{L}^{-1} \left[\frac{1}{u_{\infty}s \left(s + \frac{p}{u_{\infty}} \right)} \right] = \frac{1 - e^{-p(x/u_{\infty})}}{p} \quad (B3)$$

By use of a table of transforms and the substitution theorem, we get for the first term of equation (B2) the following:

$$\begin{aligned}
& \mathcal{L}^{-1} \left\{ \frac{\sqrt{\frac{\alpha_f}{u_\infty}} \sigma e^{-y \sqrt{u_\infty/\alpha_f} \sqrt{s+(p/u_\infty)}}}{p(s + \frac{p}{u_\infty}) \left(\sqrt{s + \frac{p}{u_\infty}} + \sigma \sqrt{\frac{\alpha_f}{u_\infty}} \right)} \right\} \\
& = \frac{e^{-p(x/u_\infty)}}{p} \left\{ \operatorname{erfc} \left[\frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] - e^{\sigma y + (\alpha_f x \sigma^2 / u_\infty)} \operatorname{erfc} \left[\sigma \sqrt{\frac{\alpha_f x}{u_\infty}} + \frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] \right\} \quad (B4)
\end{aligned}$$

Thus, equations (B4) and (B3) combine to give the temperature distribution function in the first transformed plane as

$$\begin{aligned}
\frac{\bar{\theta}}{\theta_c} &= \frac{e^{-p(x/u_\infty)}}{p} \left\{ \operatorname{erfc} \left[\frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] - e^{\sigma y + (\alpha_f x \sigma^2 / u_\infty)} \operatorname{erfc} \left[\sigma \sqrt{\frac{\alpha_f x}{u_\infty}} + \frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] \right\} \\
&\quad + \frac{1 - e^{-p(x/u_\infty)}}{p} \quad (B5)
\end{aligned}$$

Now equation (B5) must be inverted so that p is mapped back into time t . The inverse of the third term in equation (B5) is

$$\mathcal{L}^{-1} \left[\frac{1 - e^{-p(x/u_\infty)}}{p} \right] = 1 - u \left(t - \frac{x}{u_\infty} \right) \quad (B6)$$

where $u[t - (x/u_\infty)]$ is the unit step function.

The inverse of the first term of equation (B5) yields the following:

$$\mathcal{L}^{-1} \left\{ \frac{e^{-p(x/u_\infty)}}{p} \operatorname{erfc} \left[\frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] \right\} = u \left(t - \frac{x}{u_\infty} \right) \operatorname{erfc} \left[\frac{y}{2} \sqrt{\frac{u_\infty}{\alpha_f x}} \right] \quad (B7)$$

In attempting to invert the second term of equation (B5), it is advantageous to add and subtract $y^2 u_\infty / 4 \alpha_f x$ to the exponent $\sigma y + (\alpha_f x \sigma^2 / u_\infty)$ in order to get e to a power which is the square of the argument of the complimentary error function it multiplies, so as to

easily perform the inverse transform. Doing this and then defining for convenience yield

$$c = \frac{h_c}{r k_f} + \frac{y}{2r} \frac{u_\infty}{\alpha_f x}$$

and

$$d = r \sqrt{\frac{\alpha_f x}{u_\infty}}$$

The second term of equation (B5) becomes

$$\left(\frac{-e^{-y^2 u_\infty / 4 \alpha_f x}}{p} \right) \left\{ e^{-p(x/u_\infty)} e^{d^2(p+c)} \operatorname{erfc}[d(p+c)] \right\}$$

This result can be inverted by using the convolution theorem. For the first factor we have

$$\mathcal{L}^{-1} \left[\frac{-e^{-y^2 u_\infty / 4 \alpha_f x}}{p} \right] = -e^{-y^2 u_\infty / 4 \alpha_f x} \quad (B8)$$

The inverse of the second factor is found by simultaneously employing the substitution and translation properties of the transform as

$$\mathcal{L}^{-1} \left\{ e^{-p(x/u_\infty)} e^{d^2(p+c)} \operatorname{erfc}[d(p+c)] \right\} = \frac{u \left(t - \frac{x}{u_\infty} \right) e^{-\left\{ \left[\left(t - \frac{x}{u_\infty} \right)^2 / 4d^2 \right] + c[t - (x/u_\infty)] \right\}}{\sqrt{\pi} d} \quad (B9)$$

By using equations (B9) and (B8) and the convolution theorem, the inverse of the second term of equation (B5) becomes, using λ as a dummy variable for time and denoting the

entire second term as II, the following

$$\mathcal{L}^{-1}_{p \rightarrow t}[\text{II}] = - \frac{e^{-u_\infty y^2/4\alpha_f x} u\left(t - \frac{x}{u_\infty}\right)}{\sqrt{\pi} d} \int_{x/u_\infty}^t e^{-\left\{\left[\left(\lambda - \frac{x}{u_\infty}\right)^2/4d^2\right] + c[\lambda - (x/u_\infty)]\right\}} d\lambda \quad (B10)$$

The integration is most easily performed if we complete the square on the exponent in the integrand by adding and subtracting the term

$$\frac{h_c^2 \alpha_f^x}{k_f^2 u_\infty} + \frac{h_c y}{k_f} + \frac{u_\infty y^2}{4\alpha_f x}$$

Equation (B10) becomes, when c and d are replaced by their equivalents,

$$\mathcal{L}^{-1}_{p \rightarrow t}[\text{II}] = \frac{-e^{\left[\left(h_c^2 \alpha_f^x / k_f^2 u_\infty\right) + (h_c y / k_f)\right]} u\left(t - \frac{x}{u_\infty}\right)}{\sqrt{\pi} r \sqrt{\frac{\alpha_f^x}{u_\infty}}} \times \int_{x/u_\infty}^t \exp\left(-\left\{\frac{1}{2r} \sqrt{\frac{u_\infty}{\alpha_f^x}} \left[\lambda - \frac{x}{u_\infty} + \frac{2\alpha_f^x r^2}{u_\infty} \left(\frac{h_c}{r k_f} + \frac{y u_\infty}{2r \alpha_f^x}\right)\right]\right\}^2\right) d\lambda \quad (B11)$$

This equation suggests the following variable change:

$$\beta = \frac{1}{2r} \sqrt{\frac{u_\infty}{\alpha_f^x}} \left[\lambda - \frac{x}{u_\infty} + \frac{2\alpha_f^x r^2}{u_\infty} \left(\frac{h_c}{r k_f} + \frac{y u_\infty}{2r \alpha_f^x}\right)\right] \quad (B12)$$

$$\therefore d\beta = \frac{1}{2r} \sqrt{\frac{u_\infty}{\alpha_f^x}} d\lambda \quad (B13)$$

When the quantities η , ϵ , τ , and Y defined in the analysis are used, the changed limits on the integral are as follows:

$\lambda = x/u_\infty$ corresponds to

$$\beta = \eta + Y \quad (B14)$$

$\lambda = t$ corresponds to

$$\beta = \epsilon(\tau - 1) + \eta + Y \quad (B15)$$

When equations (B12) to (B15) are used, equation (B11) becomes

$$\mathcal{L}^{-1}_{p \rightarrow t}[\Pi] = -e^{2\eta Y + \eta^2} u(\tau - 1) \left(\frac{2}{\sqrt{\pi}} \right) \int_{\eta + Y}^{\epsilon(\tau - 1) + \eta + Y} e^{-\beta^2} d\beta \quad (B16)$$

But this, by the definition of the error function, reduces to

$$\mathcal{L}^{-1}_{p \rightarrow t}[\Pi] = -e^{2\eta Y + \eta^2} u(\tau - 1) \{ \operatorname{erf}[\epsilon(\tau - 1) + \eta + Y] - \operatorname{erf}[\eta + Y] \} \quad (B17)$$

After changing to the variables η , Y , ϵ , and τ in equations (B7) and (B6) and adding equations (B6), (B7), and (B17), we arrive at the nondimensional temperature excess distribution in the physical plane given by equation (14) of the analysis.

APPENDIX C

SOME DETAILS OF SOLUTION FOR CASE WHERE INLET TEMPERATURE IS STEPPED AFTER A STEADY STATE HAS BEEN REACHED

When fluid at temperature T_∞ is flowing over the plate of figure 1 (coolant temperature being T_c) in the steady state and then the inlet temperature is abruptly changed to T_0 , the mathematical statement of the problem is given by equations (1), (22), (23), (4), and (5).

A solution of the following form (eq. (24)) is sought:

$$\theta(x, y, t) = \varphi(x, y, t) + \theta_{ss}(x, y) \quad (24)$$

where θ_{ss} is the function given by equation (19) and satisfies the following conditions:

$$\left. \begin{array}{l} x = 0 \quad y > 0 \quad \theta_{ss} = 0 \\ y \rightarrow \infty \quad x > 0 \quad \theta_{ss} \rightarrow 0 \\ y = 0 \quad x > 0 \quad \frac{\partial \theta_{ss}}{\partial y} = \frac{h_c}{k_f} (\theta_{ss} - \theta_c) \end{array} \right\} \quad (C1)$$

Substituting equation (24) into (1) (noting that θ_{ss} satisfies the steady-state version of (1)) gives the partial differential equation for φ as

$$\frac{\partial \varphi}{\partial t} + u_\infty \frac{\partial \varphi}{\partial x} = \alpha_f \frac{\partial^2 \varphi}{\partial y^2} \quad (C2)$$

The side conditions on φ become

$$t = 0 \quad x > 0 \quad y > 0 \quad \varphi = 0 \quad (C3)$$

$$x = 0 \quad t > 0 \quad y > 0 \quad \varphi = T_0 - T_s = \varphi_0 \quad (C4)$$

$$y \rightarrow \infty \quad t > 0 \quad x > 0 \quad \varphi \text{ is finite} \quad (C5)$$

$$y = 0 \quad t > 0 \quad x > 0 \quad \frac{\partial \varphi}{\partial y} = \frac{h_c}{k_f} \varphi + r \frac{\partial \varphi}{\partial t} \quad (C6)$$

Now the following transformation is introduced in order to force the problem posed by equations (C2) to (C6) to look like the problem given by equations (1) to (5), whose solution has already been determined:

$$\gamma(x, y, t) = \varphi(x, y, t) - \varphi_0 \quad (C7)$$

Inserting this into equation (C2) gives

$$\frac{\partial \gamma}{\partial t} + u_\infty \frac{\partial \gamma}{\partial x} = \alpha_f \frac{\partial^2 \gamma}{\partial y^2} \quad (C8)$$

The side conditions on γ become

$$t = 0 \quad x > 0 \quad y > 0 \quad \gamma = -\varphi_0 \quad (C9)$$

$$x = 0 \quad t > 0 \quad y > 0 \quad \gamma = 0 \quad (C10)$$

$$y \rightarrow \infty \quad t > 0 \quad x > 0 \quad \gamma \text{ is finite} \quad (C11)$$

$$y = 0 \quad t > 0 \quad x > 0 \quad \frac{\partial \gamma}{\partial y} = \frac{h_c}{k_f} [\gamma - (-\varphi_0)] + r \frac{\partial \gamma}{\partial t} \quad (C12)$$

Inspection of equations (C8) to (C12) shows that they are identical with equations (1) to (5) if θ_c is replaced by $-\varphi_0$. Hence, the solution function for γ is given by equation (14) with θ_c replaced by $-\varphi_0$, and the solution to the total problem of interest becomes equation (26).

APPENDIX D

DERIVATION OF QUASI-STEADY WALL FLUX EXPRESSION FOR ARBITRARILY VARYING WALL TEMPERATURE UNDER SLUG-FLOW CONDITIONS

The governing equation and side conditions, in terms of $\theta = T - T_\infty$, for this quasi-steady problem are as follows:

$$u_\infty \frac{\partial \theta}{\partial x} = \alpha_f \frac{\partial^2 \theta}{\partial y^2} \quad (D1)$$

$$x = 0 \quad y > 0 \quad \theta = 0 \quad (D2)$$

$$y \rightarrow \infty \quad x > 0 \quad \theta \text{ is finite} \quad (D3)$$

$$y = 0 \quad x > 0 \quad \theta = \theta_w(x) \quad (D4)$$

Using the Laplace transformation on equation (D1) with respect to x and also on side conditions (D3) and (D4), solving the resulting equation, and applying the boundary conditions in the transformed plane yield the transformed temperature excess distribution as

$$\bar{\theta} = \bar{\theta}_w e^{-y \sqrt{u_\infty p / \alpha_f}} \quad (D5)$$

where p is the transform variable. Following Lighthill (ref. 19) only the surface heat flux in the physical plane will be found by inverting the transformed surface heat flux. The transformed quasi-steady surface heat flux is, after some rearrangement for ease of inversion,

$$\bar{q}_w, q_s = k_f \sqrt{\frac{u_\infty}{\alpha_f}} \left[\frac{1}{\sqrt{p}} (\bar{p} \bar{\theta}_w) \right] \quad (D6)$$

Inverting equation (D6) with the aid of the convolution theorem gives the following general expression for the quasi-steady surface heat flux for wall temperature varying arbitrarily with x and time t :

$$q_{w,qs} = k_f \sqrt{\frac{u_\infty}{\pi \alpha_f}} \int_0^x \frac{1}{\sqrt{x-\xi}} \frac{\partial \theta_w}{\partial \xi} d\xi \quad (30)$$

where ξ is a dummy variable for x and the integral must be interpreted in the Stieltjes sense.

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